# XI.

# NOTE ON THE REPRESENTATION OF ORTHOGONAL MATRICES.

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If $\phi$ is an orthogonal matrix of which $-1$ is not a latent root, we may put

$$\Upsilon = \frac{1 - \phi}{1 + \phi};$$

whence follows

$$\text{tr.}\,\Upsilon = \frac{1 - \text{tr.}\,\phi}{1 + \text{tr.}\,\phi} = \frac{1 - \phi^{-1}}{1 + \phi^{-1}} = \frac{\phi - 1}{\phi + 1} = -\Upsilon;$$

and we have

$$\phi = \frac{1 - \Upsilon}{1 + \Upsilon}.$$

This is Cayley's well known representation of an orthogonal matrix in terms of a skew symmetric matrix. If $-1$ is a latent root of $\phi$, but not 1, $-\phi$ will be an orthogonal matrix of which 1 is a latent root, but not $-1$; therefore, $-\phi$ may be represented as above, giving

$$\phi = -\frac{1 - \Upsilon}{1 + \Upsilon}.$$

Thus the expression

$$\pm \frac{1 - \Upsilon}{1 + \Upsilon}$$

will, for a proper value of the skew symmetric matrix $\Upsilon$, give all orthogonal matrices except those which have as latent roots both $\pm 1$. Such of these matrices as are real, and of which the multiplicity of the latent root $-1$ is even, and, more generally, any real proper orthogonal matrix, can, I find, be represented as an exponential function of a real skew symmetric matrix.

Thus, if $\phi$ is any real proper orthogonal matrix, a real skew symmetric matrix $\theta$ can always be found such that

$$\phi = e^{\theta},$$

where $e$ denotes the base of the Napierian logarithms ; $e^{\theta}$ is of course defined as the exponential series, which is convergent, and for which an expression may be obtained by Sylvester's theorem. This function of $\theta$, for all real skew symmetric matrices, gives only real proper orthogonal matrices. Therefore by taking successively all possible real skew symmetric matrices, all possible real proper orthogonal matrices may thus be found.

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## POSTSCRIPT.

Real improper orthogonal matrices are of two kinds: of the first kind are those real improper orthogonal matrices of which unity is not a latent root, or of which the multiplicity of the latent root unity is even; real improper orthogonal matrices of the second kind are those of which the multiplicity of the latent root unity is odd.

If $\phi$ is a real improper orthogonal matrix of the first kind, a real skew symmetric matrix $\theta$ can always be found such that

$$\phi = -e^{\theta}.$$

This function of $\theta$, for all real skew symmetric matrices, gives only real improper orthogonal matrices of the first kind.

A real improper orthogonal matrix of the second kind cannot be represented as a function of any real skew symmetric matrix.

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